Computational Complexity of Arranging Music

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Domain & Motivation

Why is this problem interesting?
Arrangement

- Transcribe a piece written for instruments A to be played on instruments B
- Expand repertoire (Zelda Theme played by NSO, Canon in D on Guitar)
- Automated arrangement software
Musical Choreography

- Create a performance which fits with music
- Examples: Dancing, Cinema, Skating
- Similar to arrangement, representing moves by performer as notes on instrument
Problem Statement
Given a score consisting of $n$ instrumental parts, does there exist a valid arrangement of the piece for one instrument?
P vs NP (vs PSPACE ... )
Computational Complexity

- Decision Problems
  - How long will it take to compute?
  - \( P := \text{Solvable in Polynomial time} \)
  - \( NP := \text{Nondeterministic Polynomial Time} \)
Computational Complexity

- Reduction from A to B encodes A in B, means B at least as hard as A
- All problems in NP can be ‘reduced’ or encoded inside 3SAT
3SAT

- Does boolean formula have solution?
- Literal: Variable or not a variable
- Clause: 3 literals or’d together
- Formula: Series of clauses and’d together

\[
\left( \neg x_1 \lor x_3 \lor x_4 \right) \land \left( x_2 \lor \neg x_3 \lor x_4 \right)
\]

Literal | Literal | Literal | Clause
Summary Of Results
## Summary of Results

<table>
<thead>
<tr>
<th>Required Percentage of Notes in Arrangement</th>
<th>0%</th>
<th>33%</th>
<th>50%</th>
<th>100%</th>
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</thead>
<tbody>
<tr>
<td>NPC P</td>
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- **NPC** indicates the necessary percentage of notes in the arrangement for each category.
- **P** indicates the percentage range where the effect is observed.

- **Consonance** shows that 100% is required for consonance.
- **Finite Transition Speed** shows that 100% is required for finite transition speed.
- **Max (j≥4)-note chord** shows that 100% is required for max (j≥4)-note chord.
- **Max (j=1)-note chord** shows that 33% is the required percentage for max (j=1)-note chord.
Quantifying a “good” arrangement

How do we deem an arrangement acceptable?
Criteria for Valid Arrangement

- Must be possible to be played / performed
- Must reflect the original intent of the piece (recognizable)
- Must be pleasing to listen to / watch
Limitations on performance

- Transition speed
- Number of simultaneous notes / actions
Original Intent / Recognizable

- Must maintain entire melodies
- Must keep certain percentage of original notes
Pleasant Sounding

Consonance: Simultaneous notes (chords) allowed in certain intervals
Hardness of Consonance
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| Required Percentage of Notes in Arrangement |
| --- | --- | --- | --- | --- |
| 0% | 33% | 50% | 100% |

- **Consonance**

- **Finite Transition Speed**

- **Max (j≥4)-note chord**

- **Max (j=1)-note chord**
Consonance Requirements

- Parts are included or excluded in entirety (recognizable melody)

- Any simultaneous notes must be in consonance (pleasant sounding)

- At any given time, at least $n\% \ (0 < n < 100)$ of notes in the original song must be played (original intent)
Variable Gadgets

- Variables represented by the choice of one of two parts
- At most one part can be played from pleasant sounding requirement
- At least one part must be played from original intent requirement
True / False Literal

- We create parts which must be played (true) and must be omitted (false).

- A true literal can be created by simply having a note on its own -- which must be played in the arrangement.

- A false literal can be created by creating a measure where it is in dissonance with a true literal.
Clause Gadgets

- Clauses represented by a measure three variable parts and some true/false literals

- Sufficient true / false literals added to ensure that n% of notes being placed requires at least one variable to be played (50% depicted)
Entire 3SAT

\[(\neg x_1 \lor x_3 \lor x_4) \land (x_2 \lor \neg x_3 \lor x_4)\]

- 3SAT represented as a song, followed by a satisfying assignment
Finite Transition Speed
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- **NPC** = Required Percentage of Notes in Arrangement
- **P** = Consonance
- **Max (j≥4)-note chord**
- **Max (j=1)-note chord**
Transition Requirements

- Parts are included or excluded in entirety (recognizable melody)
- Notes or chords cannot change more frequently than a half note (playable)
- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)
Variable Gadgets

- Again variables represented by the choice of one of two parts
- By having the two parts to play notes offset by a quarter note, only one can be played without violating transition requirements.
- True / false literals used as padding to ensure at least n% notes played at any time
Variable Gadgets

- Three arrangements of the 3SAT variable selection, selecting all parts; true and X1; and true and NOT(X1).
Transition Conclusions

- Using the transition exclusion established in the variable gadget, we can create clauses.

- Just like in the consonance problem these clauses can be used to create any 3SAT, showing the transition problem to be NP-hard.
Max j-note Chord
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- **Consonance**
- **Finite Transition Speed**
- **Max (j≥4)-note chord**
- **Max (j=1)-note chord**
Max j-note Chord

- Parts are included or excluded in entirety (recognizable melody)

- Only up to j notes can be played simultaneously (playable)

- At any given time, at least $n\% \ (0 < n < 100)$ of notes in the original song must be played (original intent)

- Using reduction from X-3SAT (only 1 variable is true)
Variable Gadgets ($j \geq 1$)

- Again variables represented by the choice of one of two parts

- For $j = 1$, the same variable gadget from consonance can be used (since only one can be played).

- For higher $j$, pad the measure with additional true / false literals to ensure only one part can be played
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Clause Gadgets (j >= 4)

- Clause represented by measure with many variable / literal tracks playing simultaneously, but must choose subset of at most j.

- Pad the measure with additional true / false literals to ensure only one variable can be played and still meets minimum number of note requirement.
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- **Consonance**: 100% required
- **Finite Transition Speed**: 100% required
- **Max (\(j\geq 4\))-note chord**: 100% required
- **Max (\(j=1\))-note chord**: 33% required
Clause Gadgets ($j = 1$), $\frac{1}{3}$ notes

- For case where at least n% ($0 < n \leq 33.3$) of notes must be played:

- Clause represented by measure with 3 variable tracks playing simultaneously, but can only play 1

- Must play 1 from original intent requirement

- Still NP-hard
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$j = 1, n > \frac{1}{3}$ notes (P)

- Solvable in polynomial time via 2-coloring, with two colors: played and not played
Edge Cases
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Edge Cases

- Allowing \( n=0 \) for the requirement that \( n\% \) of notes to be played makes the problem \( P \) since you can simply play no notes.

- Likewise allowing \( n=100 \) makes the problem \( P \) since you must select all tracks and thus can check that no other conditions are violated in a linear scan through the notes played.
Fun Applications
Applications

- Creating rhythm game (Rock Band, Dance Dance Revolution) tracks is thus NP-hard
- Choreographing dance and fight scenes in music is thus NP-hard.
Questions?

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**Hardness Made Easy**

Learn when to give up the search for efficient algorithms; see connections between computational problems; solve puzzles to prove theorems; solve open problems; and write papers.

Topics: NP, PSPACE, EXPTIME, EXPSPACE, SETH, approximation, fixed parameter, games & puzzles, SIAM, key problems, gadgets, and proof styles.