

Computational Complexity of Arranging Music

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$$\neg x_1 \vee x_2 \vee x_3$$

Domain & Motivation

Why is this problem interesting?

● Arrangement

- Transcribe a piece written for instruments A to be played on instruments B



- Expand repertoire (Zelda Theme played by NSO, Canon in D on Guitar)
- Automated arrangement software

● Musical Choreography

- Create a performance which fits with music
- Examples: Dancing, Cinema, Skating
- Similar to arrangement, representing moves by performer as notes on instrument

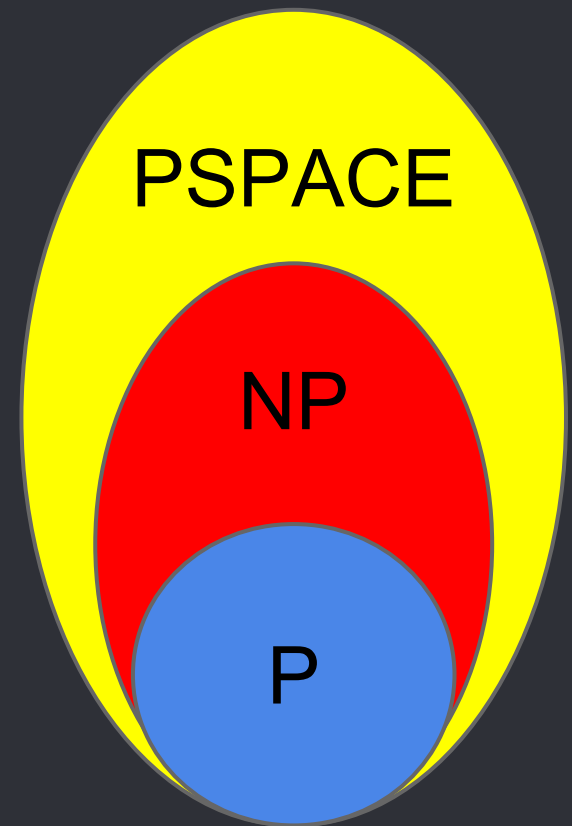


Problem Statement

P vs NP (vs PSPACE ...)

• Computational Complexity

- Decision Problems
- How long will it take to compute?
- P := Solvable in Polynomial time
- NP := Nondeterministic Polynomial Time



• Computational Complexity

- Reduction from A to B encodes A in B, means B at least as hard as A
- All problems in NP can be 'reduced' or encoded inside 3SAT



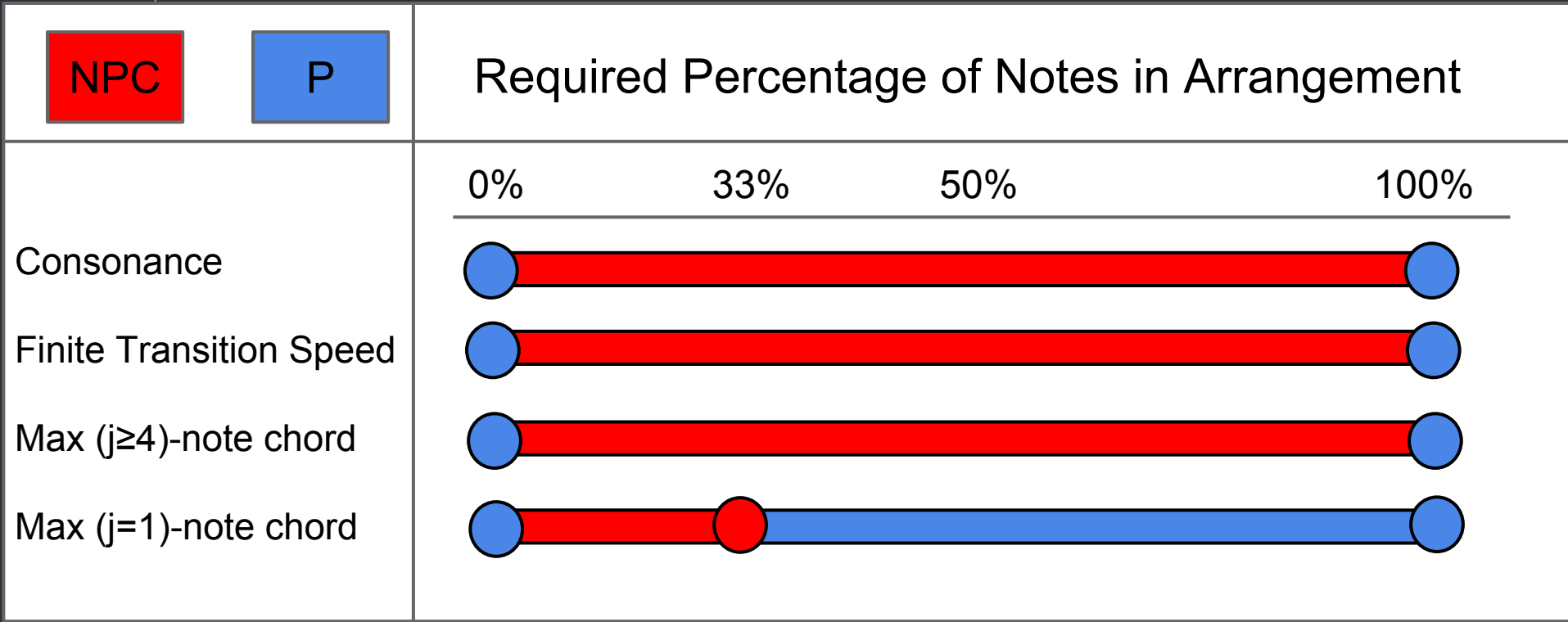
• 3SAT

- Does boolean formula have solution?
- Literal: Variable or not a variable
- Clause: 3 literals or'd together
- Formula: Series of clauses and'd together

$$\underbrace{(\neg x_1)}_{\text{Literal}} \vee \underbrace{x_3}_{\text{Literal}} \vee \underbrace{x_4}_{\text{Literal}} \wedge \underbrace{(x_2 \vee \neg x_3 \vee x_4)}_{\text{Clause}}$$

Summary Of Results

• Summary of Results



Quantifying a “good” arrangement

How do we deem an arrangement
acceptable?

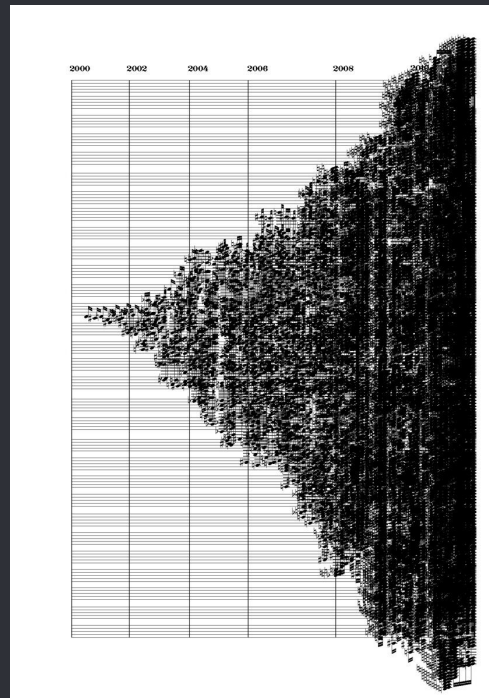
● Criteria for Valid Arrangement

- ◦ Must be possible to be played / performed
- Must reflect the original intent of the piece (recognizable)
- Must be pleasing to listen to / watch

- Limitations on performance

- Transition speed

- Number of simultaneous notes / actions



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Twinkle Twinkle Little Star

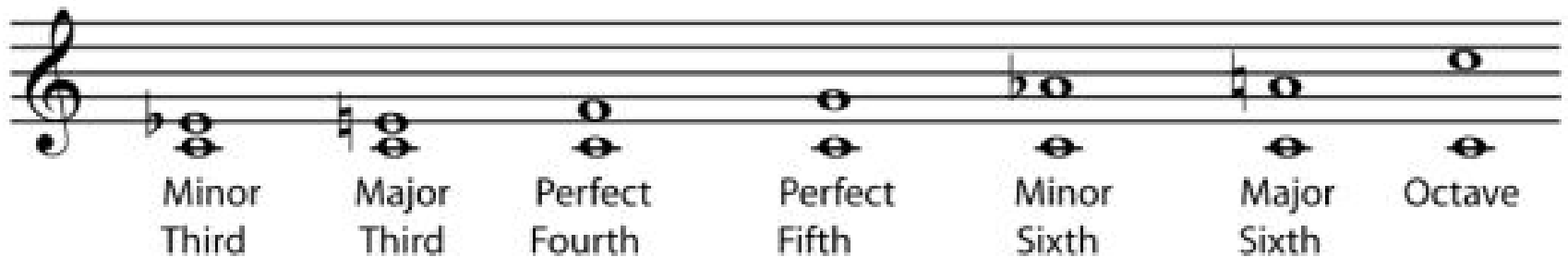
Twin - Twin - - Star

- Pleasant Sounding

- Consonance: Simultaneous notes (chords) allowed in certain intervals

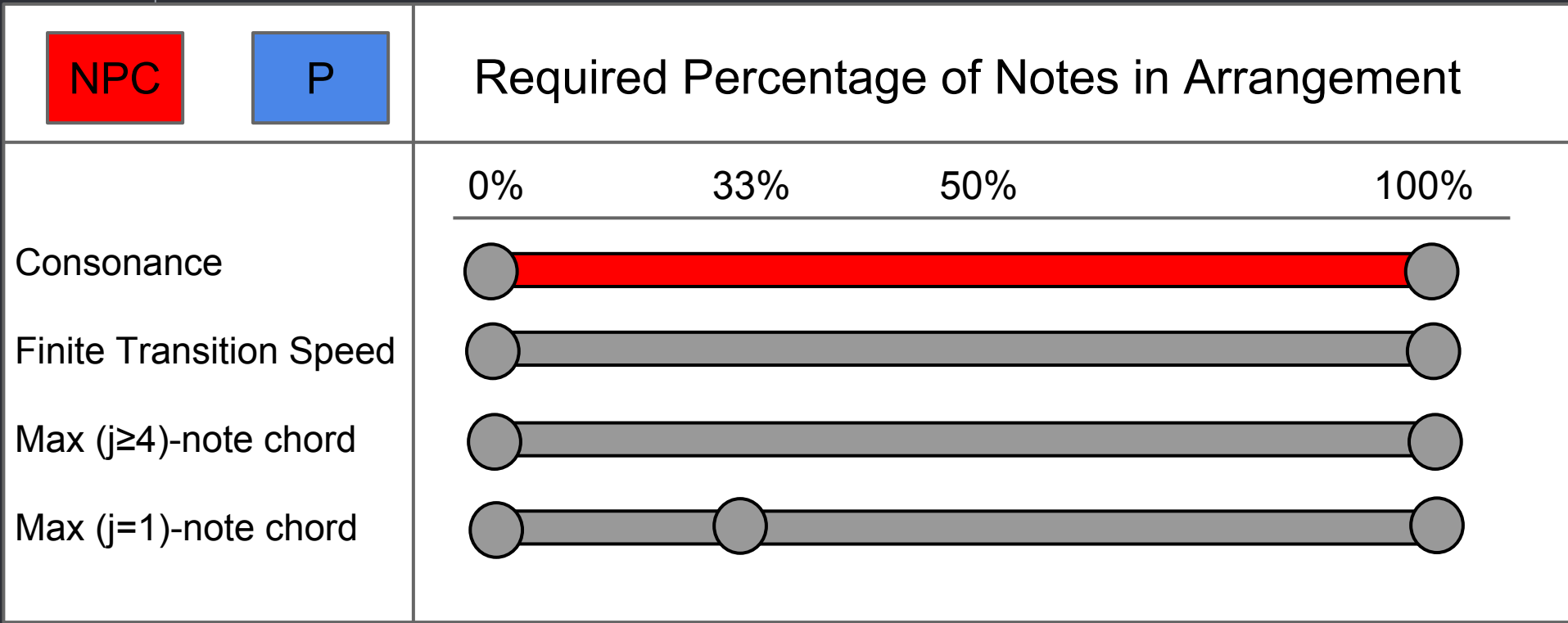


Consonant Intervals



Hardness of Consonance

- Summary of Results

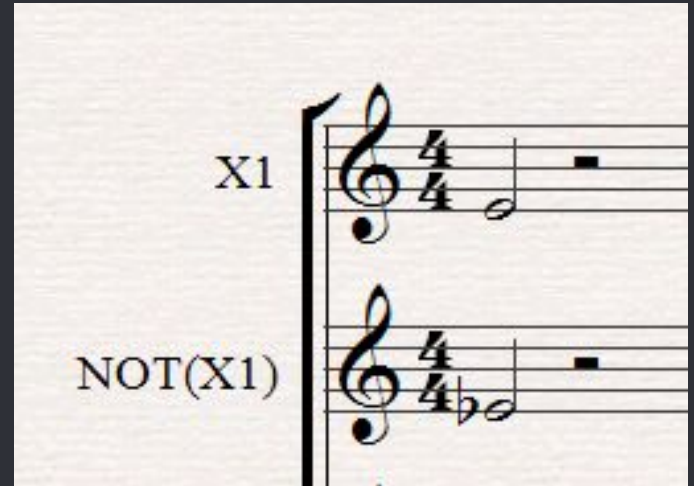


● Consonance Requirements

- Parts are included or excluded in entirety (recognizable melody)
- Any simultaneous notes must be in consonance (pleasant sounding)
- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)

• Variable Gadgets

- Variables represented by the choice of one of two parts



- At most one part can be played from pleasant sounding requirement
- At least one part must be played from original intent requirement

• True / False Literal

- We create parts which must be played (true) and must be omitted (false)



- A true literal can be created by simply having a note on its own -- which must be played in the arrangement
- A false literal can be created by creating a measure where it is in dissonance with a true literal

• Clause Gadgets

- Clauses represented by a measure three variable parts and some true/false literals
- Sufficient true / false literals added to ensure that n% of notes being placed requires at least one variable to be played (50% depicted)

The image displays a series of musical staves, each representing a clause gadget. The staves are arranged vertically and are grouped by a large right-facing curly bracket. Each staff is labeled to its left: X2, NOT(X2), X3, NOT(X3), X4, NOT(X4), and True. The first four staves (X2, NOT(X2), X3, NOT(X3)) are grouped by a bracket on the right. The fifth staff (X4) is not bracketed with the others. The last staff (True) is also not bracketed. Each staff has a 4/4 time signature. The first staff (X2) has a treble clef and a quarter note on the first line. The second staff (NOT(X2)) has a bass clef and a whole rest. The third staff (X3) has a treble clef and a whole rest. The fourth staff (NOT(X3)) has a treble clef and a quarter note on the first line. The fifth staff (X4) has a bass clef and a quarter note on the first line. The sixth staff (NOT(X4)) has a bass clef and a whole rest. The seventh staff (True) has a bass clef and a quarter note on the first line.

- Entire 3SAT

$(\neg x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$

- 3SAT represented as a song, followed by a satisfying assignment

3SAT

Billy Moses

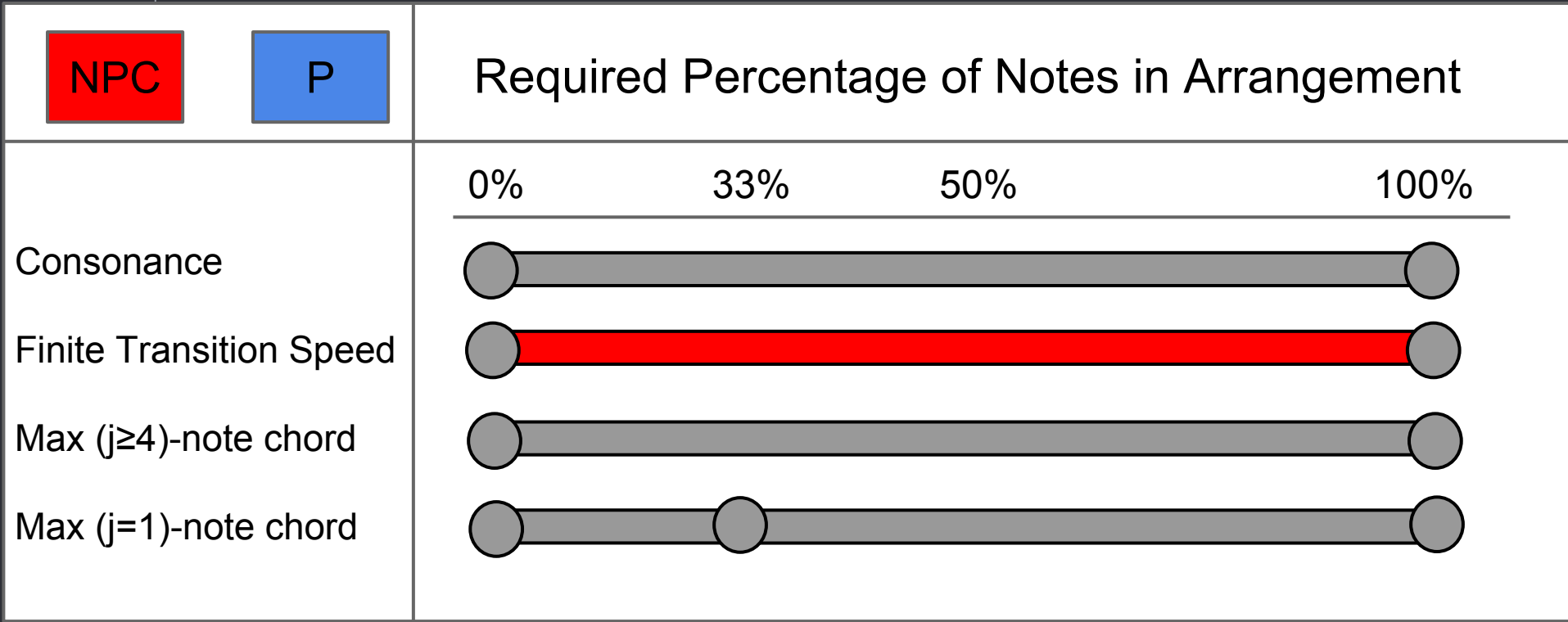


The musical score is written for 3SAT in 4/4 time. It consists of 10 staves. The first four staves are grouped by a brace on the left and labeled X1, NOT(X1), X2, and NOT(X2). The next four staves are grouped by a brace and labeled X3, NOT(X3), X4, and NOT(X4). The final staff is labeled 'True'. The notation uses treble clefs for X1, NOT(X1), X3, and NOT(X3); bass clefs for NOT(X2), NOT(X4), and 'True'; and an alto clef for X4. The key signature has one sharp (F#). The score represents the logical formula $(\neg x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$ and a satisfying assignment where x_1 is false, x_2 is true, x_3 is false, and x_4 is true.

Staff	Label	Measure 1	Measure 2	Measure 3	Measure 4	Measure 5	Measure 6	Measure 7	Measure 8
1	X1	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter note G4	Quarter rest	Quarter rest	Quarter rest
2	NOT(X1)	Quarter note Bb3	Quarter rest	Quarter rest	Quarter note G3	Quarter rest	Quarter rest	Quarter rest	Quarter rest
3	X2	Quarter rest	Quarter note G4	Quarter rest	Quarter rest	Quarter note B4	Quarter rest	Quarter note G4	Quarter note B4
4	NOT(X2)	Quarter note Bb3	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter rest
5	X3	Quarter rest	Quarter note A#4	Quarter rest	Quarter note G4	Quarter rest	Quarter rest	Quarter rest	Quarter rest
6	NOT(X3)	Quarter rest	Quarter note G4	Quarter rest	Quarter rest	Quarter note B4	Quarter rest	Quarter note G4	Quarter note B4
7	X4	Quarter rest	Quarter note G4	Quarter rest	Quarter note B4	Quarter note G4	Quarter rest	Quarter note B4	Quarter note G4
8	NOT(X4)	Quarter rest	Quarter note Bb3	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter rest	Quarter rest
9	True	Quarter rest	Quarter rest	Quarter note G4	Quarter note A4	Quarter note B4	Quarter rest	Quarter note G4	Quarter note A4

Finite Transition Speed

- Summary of Results



● Transition Requirements

- Parts are included or excluded in entirety (recognizable melody)
- Notes or chords cannot change more frequently than a half note (playable)
- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)

• Variable Gadgets

- Again variables represented by the choice of one of two parts



- By having the two parts to play notes offset by a quarter note, only one can be played without violating transition requirements.
- True / false literals used as padding to ensure at least $n\%$ notes played at any time

The image displays three musical staves, each with a treble clef and a time signature of 8/4. The first staff, labeled 'True', contains a single half note on the second line of the staff, with a long horizontal line above it indicating a duration of 8 beats. The second staff, labeled 'X1', contains a sequence of notes: a quarter note on the second line, followed by an eighth rest, a quarter note on the second line, an eighth rest, a quarter note on the second line, an eighth rest, a quarter note on the second line, and an eighth rest. The third staff, labeled 'NOT(X1)', contains a sequence of notes: a quarter note on the second line, followed by an eighth rest, a quarter note on the second line, an eighth rest, a quarter note on the second line, an eighth rest, a quarter note on the second line, and an eighth rest.

Method	Result
ArrangeAll	
ArrangeX1	
ArrangeNot	

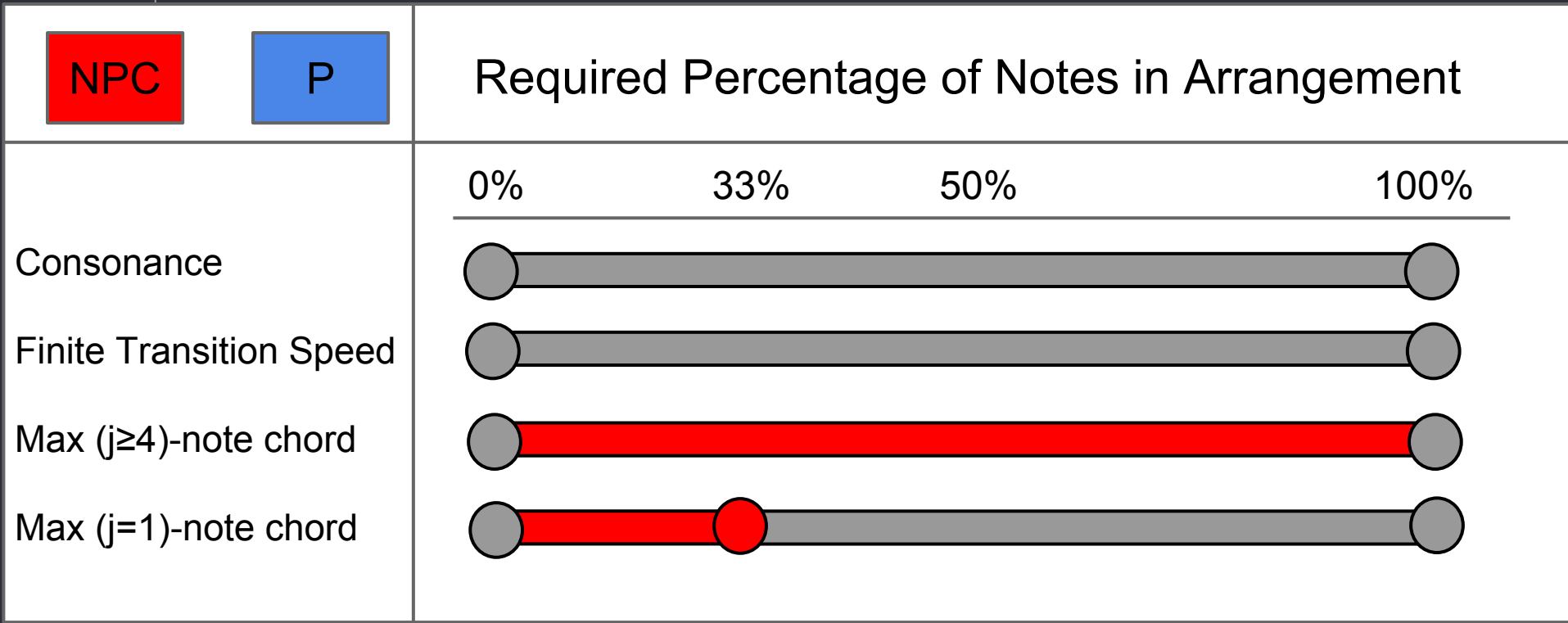
- Three arrangements of the 3SAT variable selection, selecting all parts; true and X1; and true and NOT(X1).

● Transition Conclusions

- Using the transition exclusion established in the variable gadget, we can create clauses.
- Just like in the consonance problem these clauses can be used to create any 3SAT, showing the transition problem to be NP-hard

Max j-note Chord

- Summary of Results

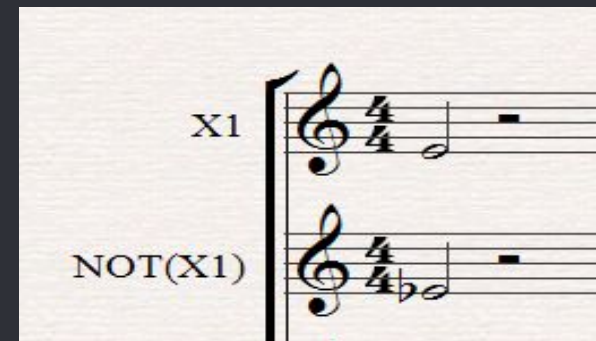


Max j-note Chord

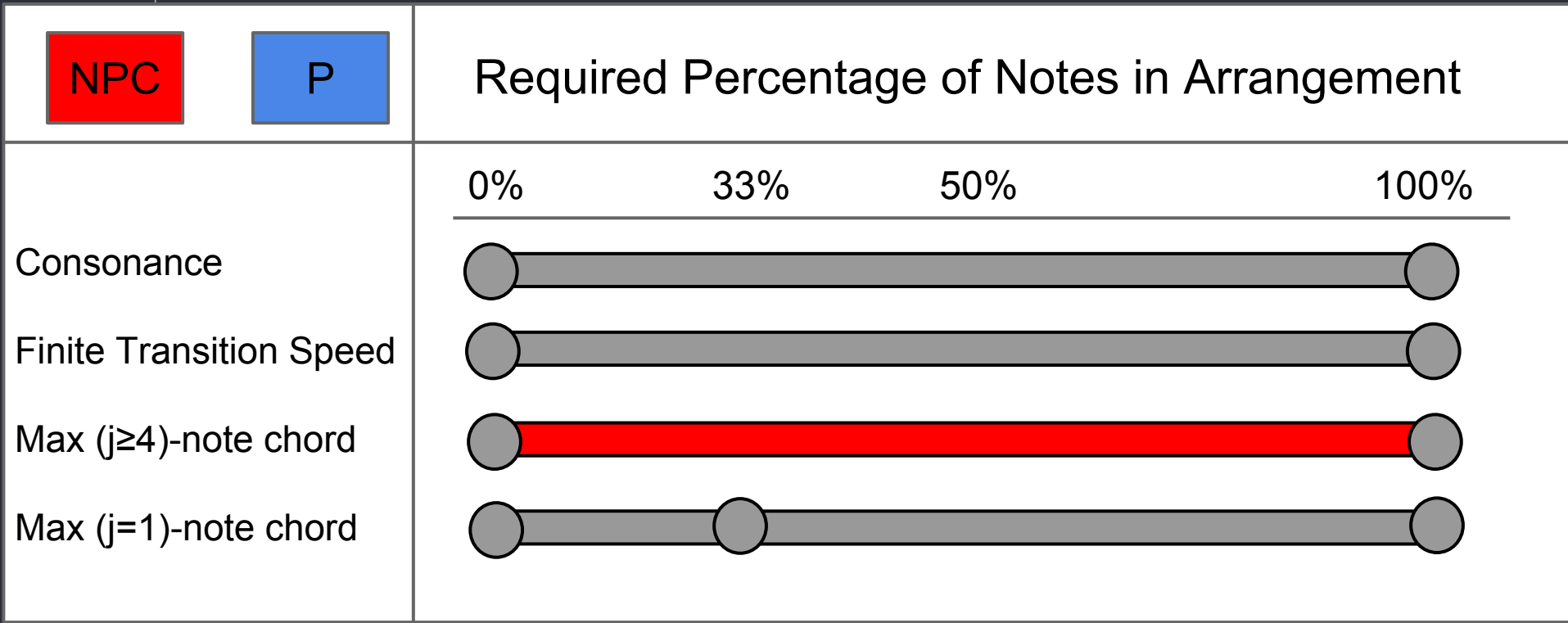
- Parts are included or excluded in entirety (recognizable melody)
- Only up to j notes can be played simultaneously (playable)
- At any given time, at least $n\%$ ($0 < n < 100$) of notes in the original song must be played (original intent)
- Using reduction from X-3SAT (only 1 variable is true)

• Variable Gadgets ($j \geq 1$)

- Again variables represented by the choice of one of two parts
- For $j = 1$, the same variable gadget from consonance can be used (since only one can be played).
- For higher j , pad the measure with additional true / false literals to ensure only one part can be played



- Summary of Results

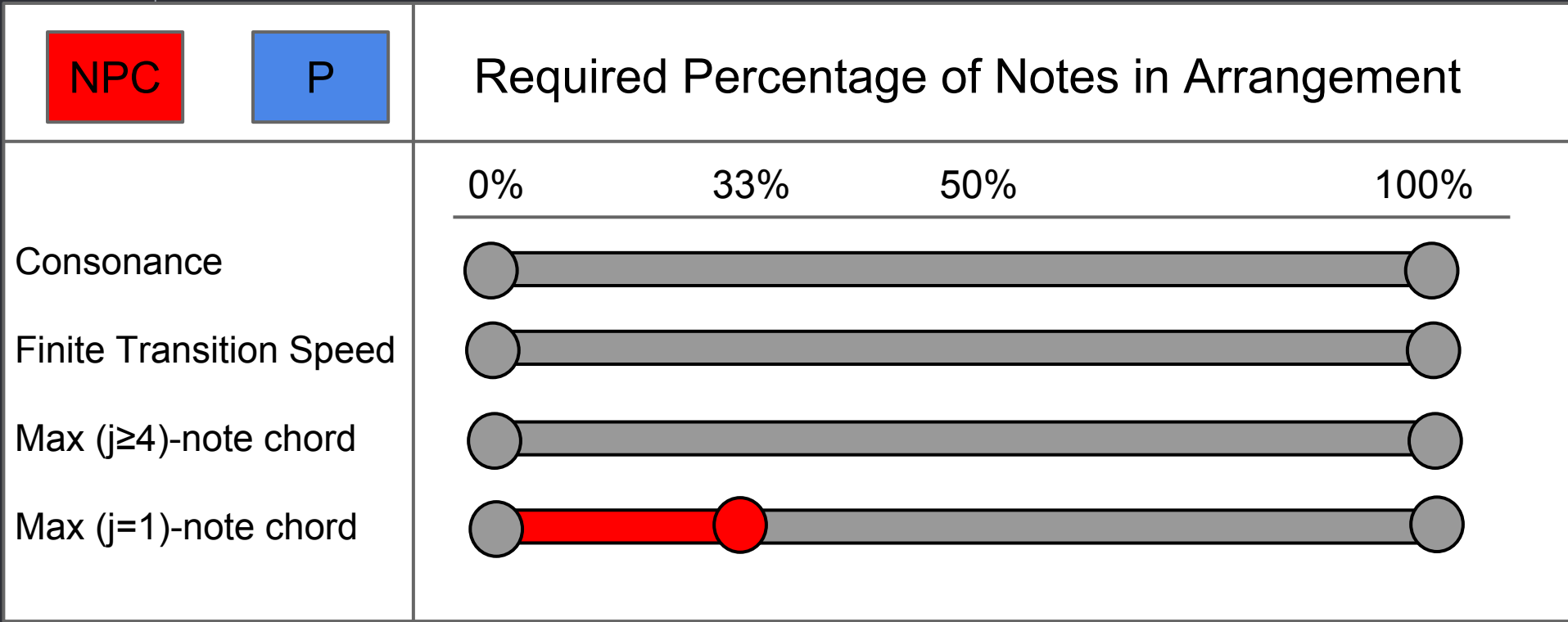


- Clause Gadgets ($j \geq 4$)

- Clause represented by measure with many variable / literal tracks playing simultaneously, but must choose subset of at most j
- Pad the measure with additional true / false literals to ensure only one variable can be played and still meets minimum number of note requirement

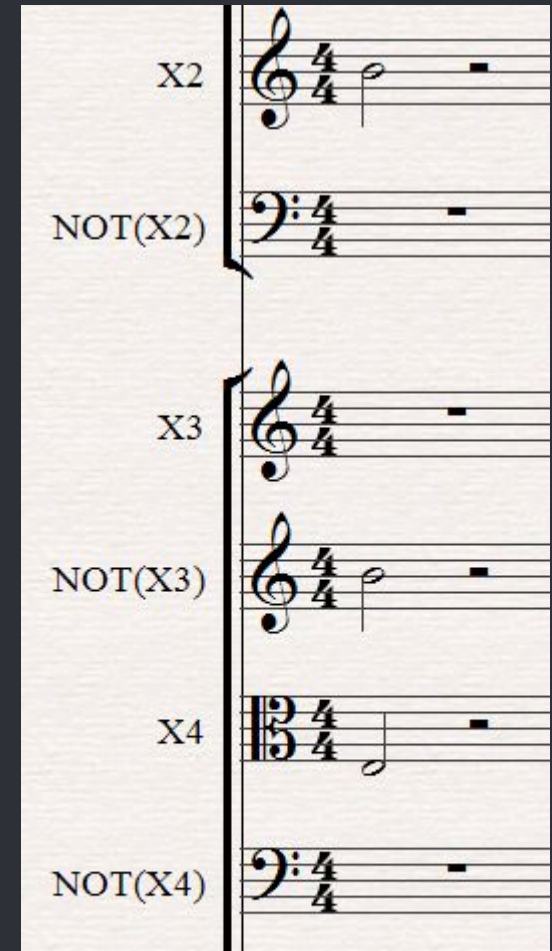
The image displays a musical score for seven staves, each representing a different clause gadget. The staves are labeled on the left as X2, NOT(X2), X3, NOT(X3), X4, NOT(X4), and True. The first four staves (X2, NOT(X2), X3, NOT(X3)) are grouped by a brace on the left. The first three staves (X2, NOT(X2), X3) are in treble clef, and the fourth (NOT(X3)) is in bass clef. The last three staves (X4, NOT(X4), True) are in bass clef. All staves have a 4/4 time signature. The notation for each staff is as follows: X2 has a quarter note on the first line; NOT(X2) has a whole rest; X3 has a whole rest; NOT(X3) has a quarter note on the first line; X4 has a quarter note on the first line; NOT(X4) has a whole rest; and True has a quarter note on the first line.

- Summary of Results



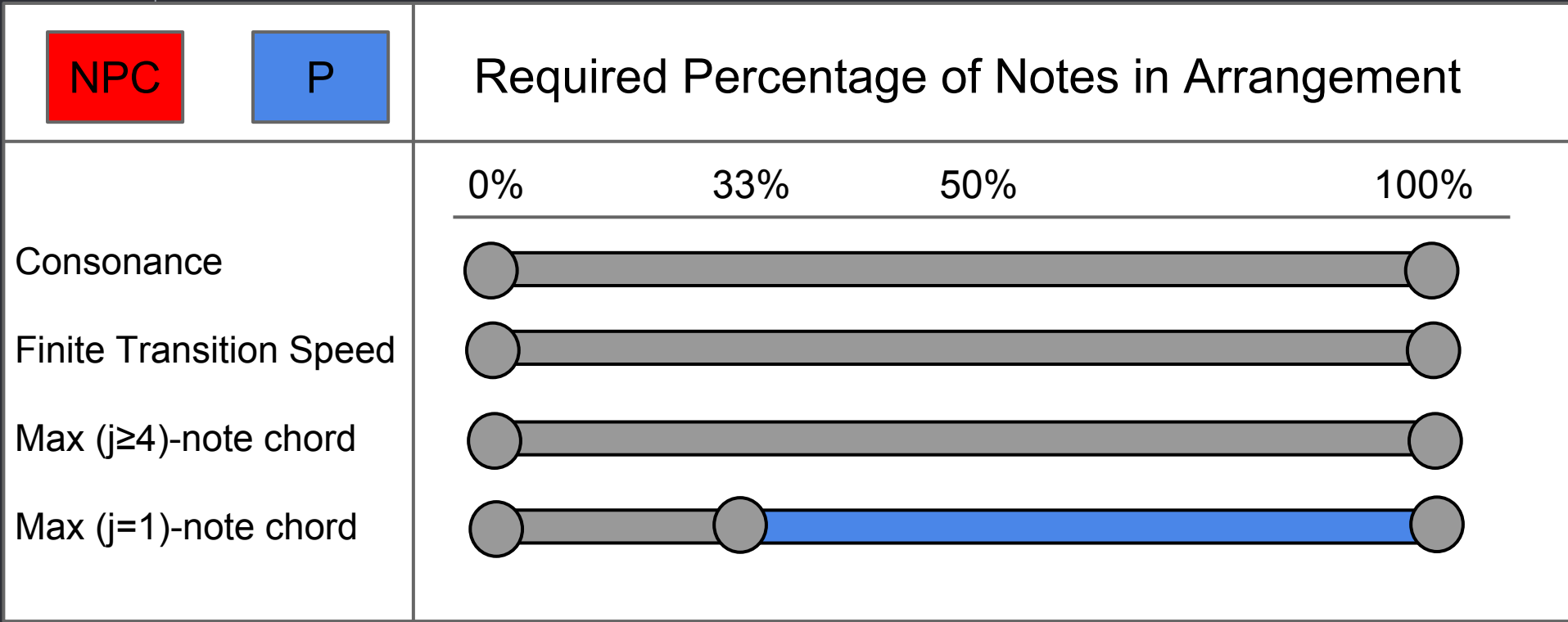
• Clause Gadgets ($j = 1$), $\frac{1}{3}$ notes

- For case where at least $n\%$ ($0 < n \leq 33.3$) of notes must be played:
- Clause represented by measure with 3 variable tracks playing simultaneously, but can only play 1
- Must play 1 from original intent requirement
- Still NP-hard

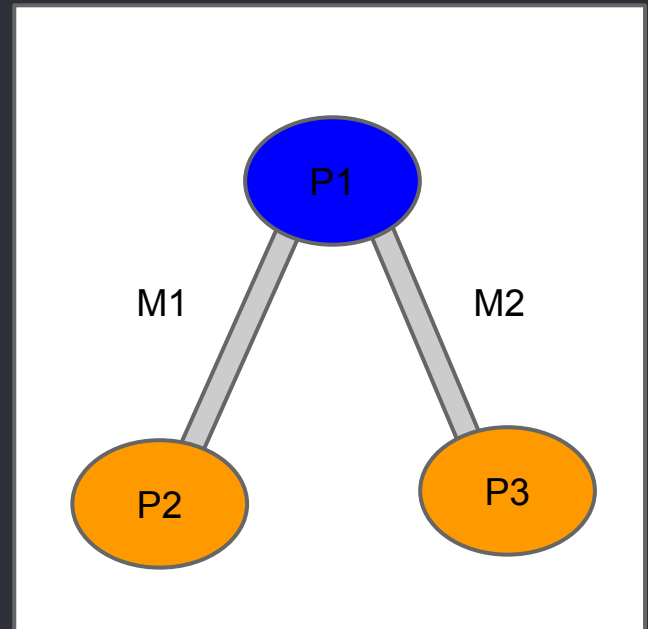


The image displays musical notation for clause gadgets, organized into three pairs of staves. Each pair is enclosed in a large curly brace on the left. The first pair consists of a treble clef staff labeled 'X2' and a bass clef staff labeled 'NOT(X2)'. The second pair consists of two treble clef staves, the top one labeled 'X3' and the bottom one labeled 'NOT(X3)'. The third pair consists of a bass clef staff labeled 'X4' and another bass clef staff labeled 'NOT(X4)'. All staves are in 4/4 time. The 'X2' staff contains a half note on G4. The 'NOT(X3)' staff contains a half note on G4. The 'X4' staff contains a half note on G2. All other staves ('NOT(X2)', 'X3', and 'NOT(X4)') contain whole rests.

- Summary of Results

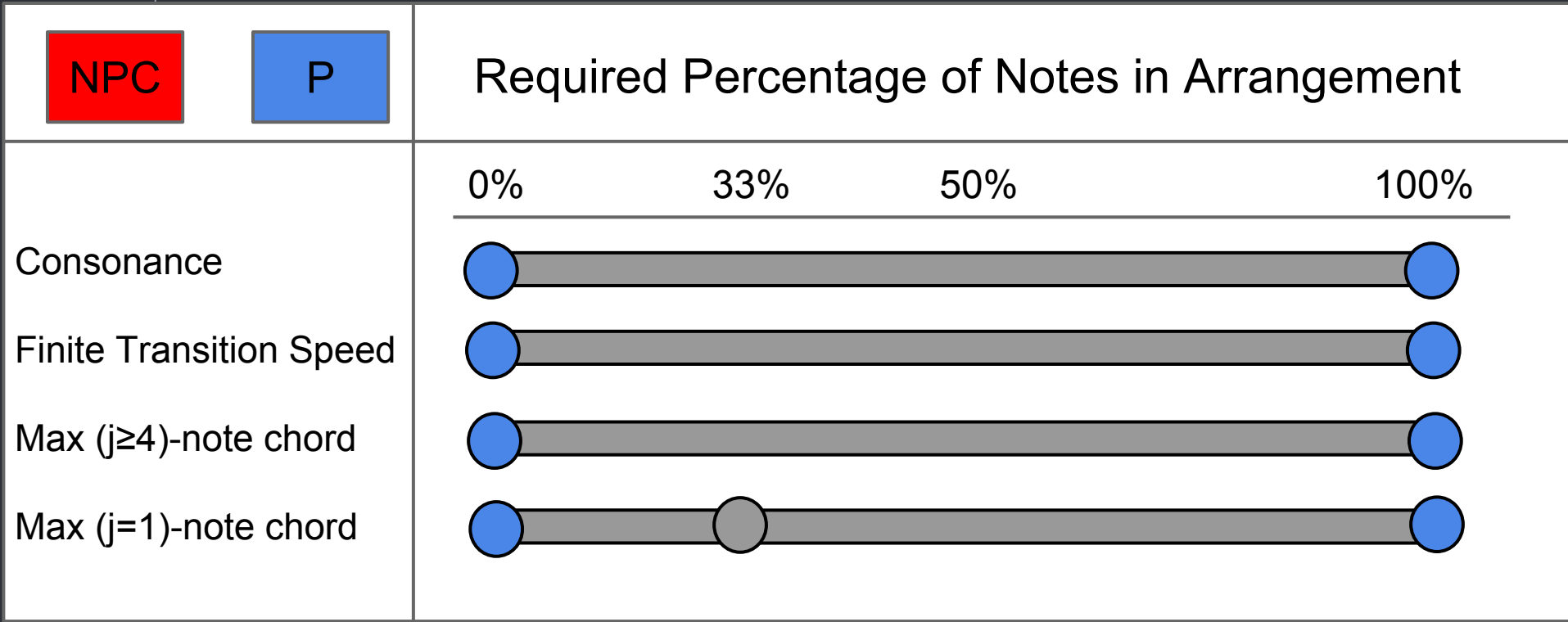


- $j = 1, n > \frac{1}{3}$ notes (P)
 - Solvable in polynomial time via 2-coloring, with two colors: played and not played



Edge Cases

- Summary of Results



Edge Cases

- Allowing $n=0$ for the requirement that $n\%$ of notes to be played makes the problem P since you can simply play no notes
- Likewise allowing $n=100$ makes the problem P since you must select all tracks and thus can check that no other conditions are violated in a linear scan through the notes played

Fun Applications

● Applications

- Creating rhythm game (Rock Band, Dance Dance Revolution) tracks is thus NP-hard
- Choreographing dance and fight scenes in music is thus NP-hard.



Handwritten musical score for two systems, each containing five staves. The notation is in treble and bass clefs, with a 3/4 time signature. The first system includes staves labeled X1, NOT(X1), X2, NOT(X2), and the second system includes X3, NOT(X3), X4, NOT(X4), and Contrabasso.

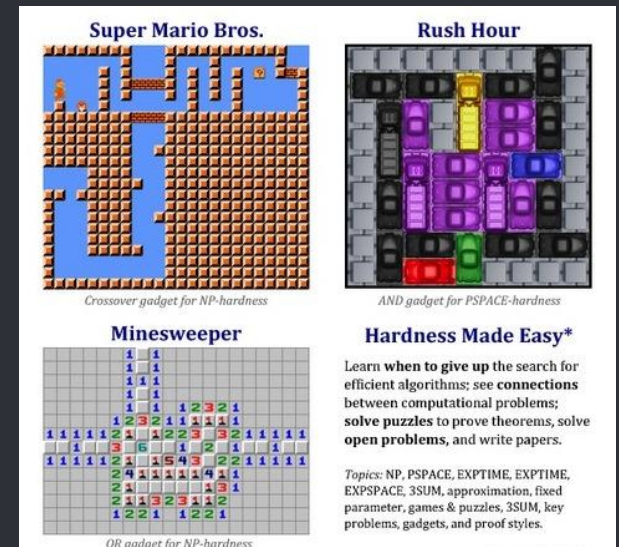
System 1:

- X1:** Treble clef, 3/4 time. Notes: Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter note G4, Quarter rest, Quarter rest, Quarter rest.
- NOT(X1):** Treble clef, 3/4 time. Notes: Quarter rest, Quarter rest, Quarter note G4, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest.
- X2:** Treble clef, 3/4 time. Notes: Quarter note G4, Quarter rest, Quarter rest, Quarter note G4, Quarter note G4, Quarter rest, Quarter rest, Quarter note G4.
- NOT(X2):** Bass clef, 3/4 time. Notes: Quarter note G2, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest.

System 2:

- X3:** Treble clef, 3/4 time. Notes: Quarter rest, Quarter rest, Quarter note G4, Quarter note G4, Quarter rest, Quarter rest, Quarter rest, Quarter rest.
- NOT(X3):** Treble clef, 3/4 time. Notes: Quarter rest, Quarter note G4, Quarter note G4, Quarter note G4, Quarter rest, Quarter note G4, Quarter note G4, Quarter note G4.
- X4:** Bass clef, 3/4 time. Notes: Quarter rest, Quarter note G2, Quarter note G2, Quarter note G2, Quarter rest, Quarter note G2, Quarter note G2, Quarter note G2.
- NOT(X4):** Bass clef, 3/4 time. Notes: Quarter note G2, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest, Quarter rest.
- Contrabasso:** Bass clef, 3/4 time. Notes: Quarter rest, Quarter rest, Quarter note G2, Quarter note G2, Quarter rest, Quarter rest, Quarter note G2, Quarter note G2.

Questions?



NPC

P

Required Percentage of Notes in Arrangement

